4) Russo-Seymour-Welsh Property

We have seen that $\mathbb{P}_{\frac{1}{2}}(\mathcal{H}_n) = \frac{1}{2}$ for all n. This quantity does not tend to 0 or 1 when $n \to +\infty$ and is actually bounded away from 0 and 1. $\rightsquigarrow$ scale-invariance property.

First shown by Russo ('78) and Seymour-Welsh ('77) and the proof has been simplified a lot since then. We present the proof from Bollobás-Riordan ('06).

Thm: (RSW property)

Let $\rho > 0$. There exists $c = c(\rho) > 0$ s.t. for all $n \geq 1$, we have

$$c \leq \mathbb{P}_{\frac{1}{2}}[\mathcal{H}(\rho n, n)] \leq 1 - c$$

where $\mathcal{H}(\rho n, n)$ denotes the horizontal crossing in a rectangle with size $\rho n \times n$.

Rmk: 1) For $\rho = 1$, we have already seen.

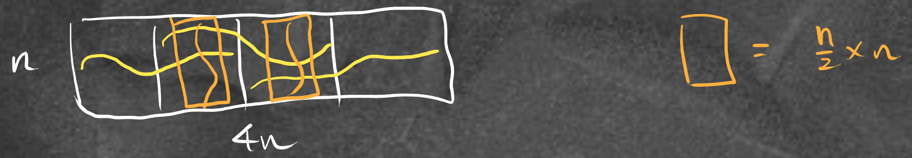
2) For $\rho < 1$, it is actually dual to $\rho > 1$ case. We only need to show for $\rho > 1$.

3) For $\rho > 1$, we say that the horizontal crossing is the "hard direction"

vertical crossing is the "easy direction"

4) It is enough to show this for one value of p. For example, if it is true for p=2.

Then,



These events together create a horizontal crossing in $4n \times n$. By the Harris inequality,

$$\mathbb{P}_{\frac{1}{2}}[\mathcal{H}(4n,n)] \geq \mathbb{P}_{\frac{1}{2}}[\mathcal{H}(2n,n)]^5 \geq c(2)^5 .$$

Proof: We are going to show the following result.

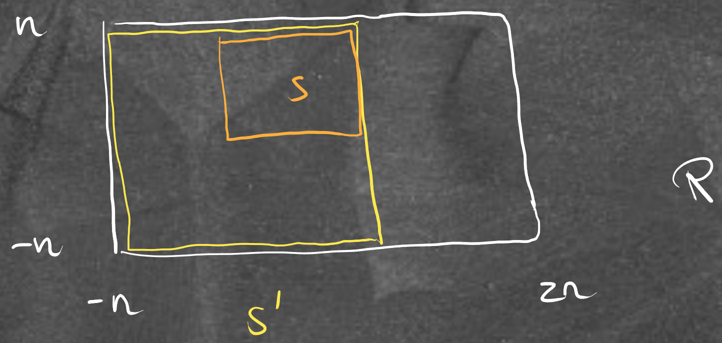
Thm: For $n \geq 1$, $\mathbb{P}_{\frac{1}{2}}[\mathcal{H}(3n,2n)] \geq \frac{1}{128}$. (Bollobás–Riordan)

Consider the following regions:

$S = [0,n] \times [0,n]$

$S' = [-n,n] \times [-n,n]$

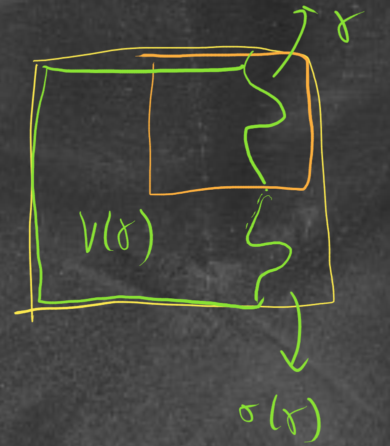
$R = [-n,2n] \times [-n,n]$



and the events $A = \mathcal{V}(S)$, $B = \{ L(S') \overset{S'}{\longleftrightarrow} R(S) \} \cap \mathcal{H}(S)$.

Write Γ for the right-most vertical crossing in S (exploration from $R(S)$ to the left).

Given a vertical path γ in S, denote $\sigma(\gamma)$ its symmetry w.r.t. $\mathbb{Z}\times\{0\}$.



Write $V(\gamma)$ for the set of edges on the left of $\gamma \cup \sigma(\gamma)$.

Note that the event $\{\Gamma=\gamma\}$ is independent of $V(\gamma)$.

Then, we find

$$\mathbb{P}_{\frac12}[B] \geq \mathbb{P}_{\frac12}[B\cap A] = \sum_{\gamma} \mathbb{P}_{\frac12}[B\cap A\cap\{\Gamma=\gamma\}]$$

$$= \sum_{\gamma} \underbrace{\mathbb{P}_{\frac12}[B \mid A\cap\{\Gamma=\gamma\}]}_{\text{measurable on edges on the right of } V(\gamma)} \mathbb{P}_{\frac12}[A\cap\{\Gamma=\gamma\}]$$

$$\geq \mathbb{P}_{\frac12}[L(S') \overset{V(\gamma)}{\longleftrightarrow} \gamma] \geq \frac14$$

$\mathbb{P}_{\frac12}(\text{left} \leftrightarrow \text{right}) \geq \mathbb{P}(\mathcal{H}(S')) = \frac12$

$\mathbb{P}_{\frac12}(L(S') \leftrightarrow \gamma) \geq \frac12 \cdot \frac12 = \frac14$

$$\geq \frac14 \sum_{\gamma} \mathbb{P}_{\frac12}[A\cap\{\Gamma=\gamma\}] = \frac14 \mathbb{P}_{\frac12}[A] = \frac18 .$$

To conclude, define B' to be the symmetric event of B.

Then,

$$\mathbb{P}_{\frac12}[\mathcal{H}(3n,2n)] \geq \mathbb{P}_{\frac12}[B'\cap B\cap V(S)] \overset{\text{Harris}}{\geq} \mathbb{P}_{\frac12}[B']\,\mathbb{P}_{\frac12}[B]\,\mathbb{P}_{\frac12}[V(S)]$$

$$= \frac{1}{128}.$$

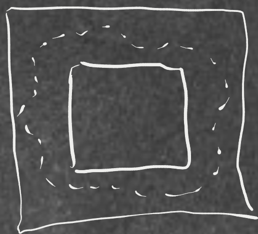
∎

<u>Consequence</u>: There exists $\alpha \in (0,\infty)$ s.t.

$$\frac{1}{2n} \leq \mathbb{P}_{\frac12}[0 \longleftrightarrow \partial\Lambda_n] \leq \frac{1}{n^\alpha} \quad \text{for all } n \geq 1.$$

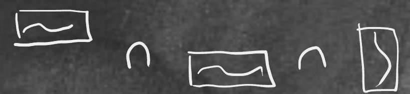
<u>Proof</u>: The lower bound was obtained in an exercise.

Let us look at the event $\{\partial\Lambda_n \overset{\Lambda_{2n}\setminus\Lambda_n}{\not\longleftrightarrow} \partial\Lambda_{2n}\}$. If this event holds, it means that there is a dual blocking circuit in the annulus $\Lambda_{2n}\setminus\Lambda_n$.



Hence, $\mathbb{P}_{\frac12}(\partial\Lambda_n \overset{\Lambda_{2n}\setminus\Lambda_n}{\not\longleftrightarrow} \partial\Lambda_{2n}) = \mathbb{P}_{\frac12}(\text{dual circuit in } \Lambda_{2n}\setminus\Lambda_n)$

$= \mathbb{P}_{\frac12}(\text{primal circuit in } \Lambda_{2n}\setminus\Lambda_n)$ (self-duality at $\frac12$)

$\geq \mathbb{P}_{\frac12}($  $)$

$\geq c(3)^4$ by RSW + Harris. (This holds for all n)

Finally,

$$\mathbb{P}_{\frac12}[0 \longleftrightarrow \partial\Lambda_n] \leq \mathbb{P}_{\frac12}[\partial\Lambda_{2^k} \overset{\Lambda_{2^{k+1}}\setminus\Lambda_{2^k}}{\longleftrightarrow} \partial\Lambda_{2^{k+1}} \text{ for all } k \text{ s.t. } 2^{k+1} \leq n]$$

$$\leq (1-c)^{\lfloor \log_2 n \rfloor} \leq \frac{1}{n^\alpha} \quad \text{for some } \alpha > 0.$$

Exercise: Consider the rectangles $R_1 = [0,n]\times[0,2n]$, $R_2 = [0,n]\times[n,3n]$, $R_3 = [0,n]\times[2n,4n]$,

$R_4 = [0,2n]\times[n,2n]$, $R_5 = [0,2n]\times[2n,3n]$.

1) Show that $\mathbb{P}_p[\mathcal{H}(n,4n)] \le 5\,\mathbb{P}_p[\mathcal{H}(n,2n)]$.

2) Deduce that $u_{2n} \le 25\, u_n^2$ where $u_n = \mathbb{P}_p[\mathcal{H}(n,2n)]$.

3) Show that either $u_n \ge \frac{1}{25}$ for all n or (u_n) decays exp. fast.

4) What happens at p_c? (Assume that we have no knowledge about p_c.)