3) Exponential Decay

3.1) Statement and proof

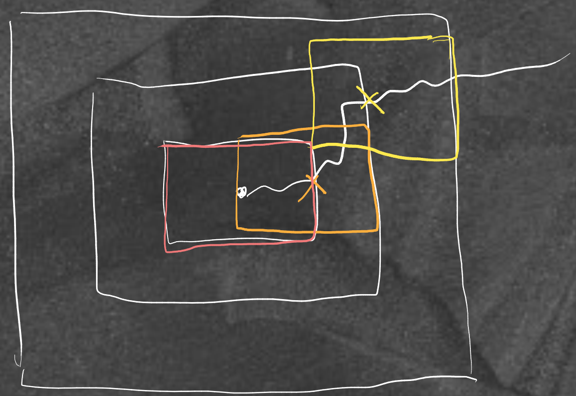
Exercise: Can we have a decay faster than exponential?

Thm: For $d \geq 1$, the following statements hold:

1) for $p < p_c$, there exists $c = c(p) > 0$ s.t. for all $n \geq 1$, $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq \exp(-cn)$.

2) For $p > p_c$, $\mathbb{P}_p[0 \leftrightarrow \infty] \geq \dfrac{p - p_c}{p(1-p_c)}$.

Rmk:



If the origin wants to be connected to $\partial\Lambda_{kL}$ then it must reach $\partial\Lambda_{mL}$ for $1 \leq m \leq k$.

Heuristically, if $c = \mathbb{P}(0 \leftrightarrow \partial\Lambda_L) < 1$, then at each level, an additional cost c is spent, giving an exponential decay.

$$\mathbb{P}(0 \leftrightarrow \partial\Lambda_{2L}) = \sum_{x \in \partial\Lambda_L} \mathbb{P}(0 \leftrightarrow x \in \partial\Lambda_L) \cdot \mathbb{P}(x \leftrightarrow \partial\Lambda_{2L} \mid 0 \leftrightarrow x \in \partial\Lambda_L)$$

$$\text{“}\leq\text{”} \sum_{x \in \partial\Lambda_L} \mathbb{P}(0 \leftrightarrow x \in \partial\Lambda_L) \cdot \underbrace{\mathbb{P}(x \leftrightarrow x + \partial\Lambda_L)}_{= \mathbb{P}(0 \leftrightarrow \partial\Lambda_L) = c}$$

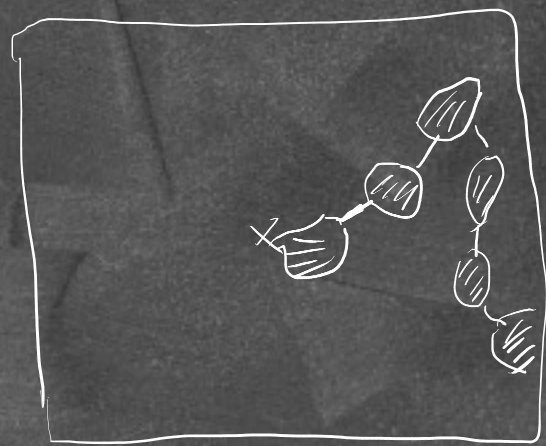
$\rightsquigarrow$ This inequality is not necessarily true.

$$= \mathbb{P}(0 \leftrightarrow \partial\Lambda_L)^2.$$

Rmk: 1) We give a more modern proof which can be extended easily to other contexts provided by Duminil-Copin-Tassion 2016. The "classical" proof by Aizenman-Barsky, Menshikov goes as follows.

(a) Find all the pivotal edges $e_1, \dots, e_k$ for $A_n = \{0 \leftrightarrow \partial \Lambda_n\}$ and ω.

(b) Estimate the size of the saucissages (highly correlated) but can be "stochastically" dominated by i.i.d. r.v.s.



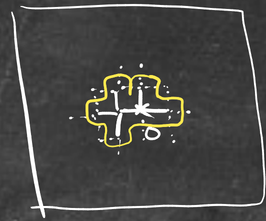
$$\mathbb{E}[N(A_n) \mid A_n] = \sum_{k \geq 1} \mathbb{P}[\underbrace{N(A_n) \geq k}_{} \mid A_n]$$

if after visiting k saussages we haven't gone far away from the origin, then we need more pivotal edges.

$$\geq \sum_{k \geq 1} \mathbb{P}(M_1 + \dots + M_k \leq n)$$

some tricks to get a linear lower bound.

2) Exploration process: how to "discover" the cluster containing 0?



Starting from the origin and sample the edges around it.

If an edge is open, we go to other endpoint and check the state of its uncovered neighbors. If an edge is closed, we stop there.

What has been discovered and what is left to discover are Independent! (Bernoulli perc.)

Moreover, we find a blocking circuit enclosing our cluster.

Proof: We follow a more modern proof by Duminil-Copin–Tassion (2016).

We define the critical parameter in a different way.

For $p \in [0,1]$ and $0 \in S \subseteq \mathbb{Z}^d$, define

$$\varphi_p(S) = p \sum_{\{x,y\} \in \Delta S} \mathbb{P}_p[0 \overset{S}{\longleftrightarrow} x]$$

dual of the blocking surface

where ΔS denotes the "edge boundary", $\Delta S := \{ \{x,y\} \in E,\ x \in S,\ y \notin S \}$.

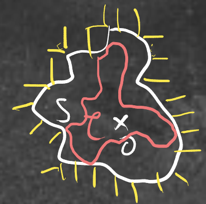
Hence, $\varphi_p(S) = \mathbb{E}[$ open edges on the edge boundary ΔS that are connected to 0 in $S]$.

Let $\tilde{p}_c := \sup\{ p \in [0,1],$ there is a finite set $0 \in S \subseteq \mathbb{Z}^d$ s.t. $\varphi_p(S) < 1 \}$.

We will show the theorem with p_c replaced with $\tilde{p}_c$, which gives automatically $p_c = \tilde{p}_c$.

(1) First, we show the exp. decay in the subcritical regime.

Let $p < \tilde{p}_c$ and consider a finite set $0 \in S \subseteq \mathbb{Z}^d$ s.t. $\varphi_p(S) < 1$.



ΔS

Let $L \geq 1$ s.t. $S \subseteq \Lambda_{L-1}$. Let $\mathcal{C} = \{ z \in S : 0 \xleftrightarrow{S} z \}$.

For $k \geq 1$, if $\{ 0 \leftrightarrow \partial \Lambda_{kL} \}$ holds, then we can find an edge $\{x,y\} \in \Delta S$ s.t.

$$\left.\begin{array}{l} *) \quad 0 \longleftrightarrow x \text{ in } S, \\ *) \quad \{x,y\} \text{ is open}, \\ *) \quad y \longleftrightarrow \partial\Lambda_{kL} \text{ in } \mathcal{C}^c. \end{array}\right\} \text{ events using disjoint edges}$$

Using the BK Inequality, we find

$$\mathbb{P}[0 \leftrightarrow \partial\Lambda_{kL}] \leq \sum_{\{x,y\} \in \Delta S} \mathbb{P}\left[\{ 0 \xleftrightarrow{S} x \} \circ \{ \{x,y\} \text{ is open} \} \circ \{ y \xleftrightarrow{\mathcal{C}^c} \partial\Lambda_{kL} \} \right]$$

$$\overset{BK}{\leq} \sum_{\{x,y\} \in \Delta S} \mathbb{P}(0 \xleftrightarrow{S} x) \cdot p \cdot \underbrace{\mathbb{P}(y \xleftrightarrow{\mathcal{C}^c} \partial\Lambda_{kL})}_{\leq \mathbb{P}(y \longleftrightarrow y + \partial\Lambda_{(k-1)L}) = \mathbb{P}[0 \leftrightarrow \partial\Lambda_{(k-1)L}]}$$

$$\leq \varphi_p(S) \cdot \mathbb{P}[0 \longleftrightarrow \partial\Lambda_{(k-1)L}].$$

By induction, we get $\quad \mathbb{P}[0 \leftrightarrow \partial\Lambda_{kL}] \leq \varphi_p(S)^k$.

(2) For the supercritical regime, we start with the following lemma.

Lemma: For $p \in [0,1]$ and $n \geq 1$,

$$\frac{d}{dp} \mathbb{P}_p[0 \longleftrightarrow \partial \Lambda_n] \geq \frac{1}{p(1-p)} \inf_{\substack{S \subseteq \Lambda_n \\ 0 \in S}} \varphi_p(S) \cdot \left(1 - \mathbb{P}_p[0 \longleftrightarrow \partial \Lambda_n]\right).$$

First, we show how to deduce the result from the lemma.

Let $n \geq 1$ and let $f(p) = \mathbb{P}_p[0 \longleftrightarrow \partial \Lambda_n]$. When $p > \tilde{p}_c$, we know that $\varphi_p(S) \geq 1$ for all S.

Thus,
$$f'(p) \geq \frac{1}{p(1-p)} [1 - f(p)] \quad \text{for all } p \in (\tilde{p}_c, 1].$$

$$\Rightarrow \quad -\frac{d}{dp} \ln(1 - f(p)) \geq \frac{1}{p(1-p)}, \quad \forall p \in (\tilde{p}_c, 1].$$

$$\Rightarrow \quad -\ln(1-f(p)) + \ln(1 - f(\tilde{p}_c)) \geq \ln \frac{p}{\tilde{p}_c} - \ln \frac{1-p}{1-\tilde{p}_c}, \quad \forall p \in (\tilde{p}_c, 1]$$

$$\Rightarrow \quad \frac{1 - f(\tilde{p}_c)}{1 - f(p)} \geq \frac{p}{\tilde{p}_c} \cdot \frac{1-\tilde{p}_c}{1-p}, \quad \forall p > \tilde{p}_c$$

Since $f(\tilde{p}_c) \geq 0$, we get $f(p) \geq 1 - \frac{\tilde{p}_c}{1-\tilde{p}_c} \cdot \frac{1-p}{p} = \frac{p - \tilde{p}_c}{p(1-\tilde{p}_c)}$.

Taking the non-increasing limit of $\mathbb{P}_p[0 \longleftrightarrow \partial \Lambda_n]$ when $n \to \infty$, we can conclude. ▨

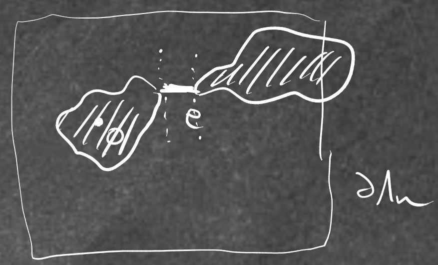
Now, let us show the lemma.

Proof: By Russo's formula,

$$\frac{d}{dp} \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] = \sum_{e \in E(\Lambda_n)} \mathbb{P}_p(e \text{ is pivotal for } \{0 \leftrightarrow \partial\Lambda_n\})$$

Note that if $\omega \in \Omega_n$ is a configuration s.t. $e \in E(\Lambda_n)$ is pivotal, it is equivalent to saying that

$\omega^e \in \{0 \leftrightarrow \partial\Lambda_n\}$, $\omega_e \notin \{0 \leftrightarrow \partial\Lambda_n\}$, where

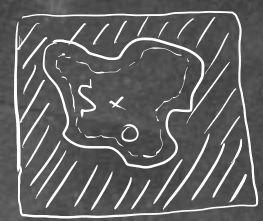


ω^e and ω_e stand for the configurations that coincide with ω outside of e and $\omega^e(e) = 1$, $\omega_e(e) = 0$.

Thus, $$\mathbb{P}_p(e \text{ is pivotal}) = \frac{1}{1-p} \mathbb{P}_p(e \text{ is pivotal}, 0 \not\leftrightarrow \partial\Lambda_n).$$

Defin the random set $\mathcal{S} := \{x \in \Lambda_n : x \not\leftrightarrow \partial\Lambda_n\}$

(Exercise: talk about the "exploration process".)



Note that $\{0 \not\leftrightarrow \partial\Lambda_n\} = \{\mathcal{S} \text{ is a subset containing the origin } 0\}$.

Hence,

$$\frac{d}{dp} \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] = \frac{1}{1-p} \mathbb{P}_p(e \text{ is pivotal}, 0 \not\leftrightarrow \partial\Lambda_n)$$

$$= \frac{1}{1-p} \sum_{0 \in S \subseteq \Lambda_n} \mathbb{P}_p(e \text{ is pivotal}, \mathcal{S} = S)$$

Moreover, on the event $\{\mathcal{S}=S\}$, the only possible pivotal edges are in ΔS. So we find

$$\mathbb{P}_p(e \text{ is pivotal}, \mathcal{S}=S) = \sum_{\{x,y\}\in\Delta S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} x, \mathcal{S}=S)$$

$$= \sum_{\{x,y\}\in\Delta S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} x)\, \mathbb{P}_p(\mathcal{S}=S)$$

($\{\mathcal{S}=S\}$ depends on S^c whereas $\{0 \overset{S}{\longleftrightarrow} x\}$ depends on S)

$$= \varphi_p(S)\cdot \mathbb{P}_p(\mathcal{S}=S)$$

In conclusion, we find

$$\frac{d}{dp}\mathbb{P}_p[0\longleftrightarrow \partial\Lambda_n] = \frac{1}{1-p}\sum_{0\in S\subseteq\Lambda_n} \mathbb{P}_p(e \text{ is pivotal}, \mathcal{S}=S)$$

$$= \frac{1}{1-p}\sum_{0\in S\subseteq\Lambda_n} \varphi_p(S)\cdot \mathbb{P}_p(\mathcal{S}=S)$$

$$\geq \frac{1}{1-p}\left(\inf \varphi_p(S)\right)\cdot \mathbb{P}_p[0 \not\longleftrightarrow \partial\Lambda_n].$$

$\blacksquare$

Consequence: Thm (Kesten, '80): $p_c(\mathbb{Z}^2)=\frac{1}{2}$. (We had already $p_c\geq\frac{1}{2}$ by showing $\theta(\frac{1}{2})=0$.)

Proof: For $p<p_c$, we have $\mathbb{P}_p(\mathcal{H}_n) \leq n\cdot \mathbb{P}_p(0\longleftrightarrow\partial\Lambda_n) \leq n\cdot e^{-cn} \longrightarrow 0$.

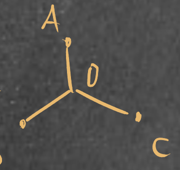
Since $\mathbb{P}_{\frac{1}{2}}(\mathcal{H}_n)=\frac{1}{2}$, we get $p_c\leq\frac{1}{2}$. $\blacksquare$

Exercise: For a planar periodic graph G and bond percolation, show that $p_c(G) + p_c(G^*) = 1$.

(Hint: uniqueness + exp. decay to make the heuristics from the previous section rigorous.)

Exercise: Find p_c for the bond percolation on $\mathbb{T}$ = triangular lattice.

1) Consider $G =$ 

, $G' =$ 

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Show that $\mathbb{P}_{p,G}$ and $\mathbb{P}_{p',G'}$ can be coupled together (in the sense that the connection properties are preserved) if $p' = 1-p$ and p satisfies $p^3 - 3p + 1 = 0$.

2) Using the exponential decay to show that p_c is given by the unique solution to $p^3 - 3p + 1 = 0$. Deduce that $p_c(\mathbb{T}) = 2\sin\left(\frac{\pi}{18}\right)$.

3.2) Size of the Finite Cluster containing 0

Usually in samplings / simulations, we are not able to look at a scale with infinitely many edges. Thus, it is interesting to understand how much we are far away from the "reality" when we look at a large scale n.

For example, the exponential decay at $p<p_c$ states that

$$\mathbb{P}_p[0 \longleftrightarrow \partial \Lambda_n] \leq \exp(-\psi(p)\cdot n) \quad \text{for some } \psi(p)>0.$$

We can actually prove a similar result for the size of the cluster containing the origin.

Exercise: Let C be the cluster containing the origin 0. Show that for $p<p_c$,

we have $\mathbb{P}_p(|C| \geq n) \leq e^{-\psi(p)\cdot n^{1/d}}$ for some $\psi(p)>0$.

(Harder) Show that $\mathbb{P}_p(|C| \geq n) \leq e^{-\lambda(p)\cdot n}$ for some $\lambda(p)>0$.

(Hint: Show that $|C|$ has an exponential moment starting with the relation

$$\mathbb{E}[|C|^n] \leq A_n \, \mathbb{E}[|C|]^{2n-1} \quad \text{where } A_n = (2n-3)!! \,, \; A_1 = 1.$$

More precisely, show and use the following relations

$$\mathbb{E}_p[|C|] = \sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(0 \longleftrightarrow x) < \infty \quad \text{and} \quad \mathbb{E}_p[|C|^n] = \sum_{x_1,\dots,x_n\in\mathbb{Z}^d} \mathbb{P}_p(0 \leftrightarrow x_1 \leftrightarrow \dots \leftrightarrow x_n).$$

3.3) Correlation length

Consider a bond percolation on $\mathbb{Z}^d$. Let $p<p_c$. Write $e_1=(1,0,\dots,0)$.

We clearly have $\mathbb{P}_p[0\leftrightarrow(n+m)e_1]\geq\mathbb{P}_p[0\leftrightarrow ne_1]\,\mathbb{P}_p[0\leftrightarrow me_1]$.

The subadditive lemma implies that $\xi(p):=\left(-\frac{1}{n}\ln\mathbb{P}_p[0\longleftrightarrow ne_1]\right)^{-1}$ exists and that

$\mathbb{P}_p[0\longleftrightarrow ne_1]\leq\exp(-n/\xi(p))$. The lower bound is the same up to a subexponential correction (more detail to come later).

The quantity $\xi(p)$ is called the correlation length (in direction e_1).

Rmk: 1) Intuitively, it is the distance order at which we still have a "good chance" to observe a connection. (path connection $\leftrightarrow$ correlation)

2) $\xi(p)<\infty$ for $p<p_c$.

3) The correlation length needs to be the same in all directions.

Let us carry on the study of the behavior of $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$.

Note that if $0 \leftrightarrow \partial\Lambda_{n+m}$, then there exists $x \in \partial\Lambda_n$ s.t. $0 \leftrightarrow x$ and $x \leftrightarrow x + \partial\Lambda_m$ hold disjointly. Thus,

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{n+m}] \leq |\partial\Lambda_n| \, \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \, \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_m], \qquad |\partial\Lambda_n| = 2d\,(2n+1)^{d-1}.$$

Exercise: Find a polynomial $f(n)$ s.t. $\left(\ln f(n) + \ln \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]\right)_{n \geq 1}$ satisfies the subadditive condition. Then, deduce that

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \geq \frac{1}{f(n)} \exp(-\varphi_1(p)\, n) \quad \text{for some } \varphi_1(p) > 0, \ p < p_c.$$

For the other bound, we may define the following auxiliary probability

$$\begin{aligned} \varphi_n &:= \mathbb{P}_p[0 \overset{\Lambda_n}{\longleftrightarrow} \text{one of the face of } \partial\Lambda_n] \\ &= \mathbb{P}_p[0 \overset{\Lambda_n}{\longleftrightarrow} x \in \partial\Lambda_n \text{ with } x_1 = n]. \end{aligned}$$

Then, $\quad \varphi_n \leq \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq 2d \cdot \varphi_n.$

Fix $x \in \partial\Lambda_n$. Let k s.t. $x_k = n$ (by symmetry).

Consider $U_x = \{ 0 \xleftrightarrow{\Lambda_n} x \}$, $V_x = \{ x \xleftrightarrow{x+\Lambda_m} y \in \partial(x+\Lambda_m),\ y_k = n+m \}$.

↳ Exercise: Why do we add the condition " in Λ_n " ?

Then, $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{n+m}] \geq \mathbb{P}_p[U_x \cap V_x] \overset{FKG}{\geq} \mathbb{P}_p(U_x)\underbrace{\mathbb{P}_p(V_x)}_{= q_m}$.

Moreover, since

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq \sum_{x \in \partial\Lambda_n} \mathbb{P}_p[0 \xleftrightarrow{\Lambda_n} x],$$

we can find a vertex $x \in \partial\Lambda_n$ s.t. $\mathbb{P}_p[0 \xleftrightarrow{\Lambda_n} x] \geq \frac{1}{|\partial\Lambda_n|}\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$.

Thus, $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_{n+m}] \geq \frac{1}{2d\cdot|\partial\Lambda_n|}\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]\,\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_m]$.

Exercise: Find a polynomial $g(n)$ s.t.

$$\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n] \leq \frac{1}{g(n)} \exp(-\varphi_2(p)\cdot n) \quad \text{for some } \varphi_2(p) > 0,\ p < p_c.$$

Check that $\varphi_1(p) = \varphi_2(p) =: \varphi(p)$.

Exercise: Deduce similar lower/upper bounds with a polynomial correction for $\mathbb{P}_p[0 \longleftrightarrow ne_1]$.

Hence, $\mathbb{P}_p[0 \longleftrightarrow ne_1]$ behaves like $\exp(-n/\xi(p))$ with $\xi(p)^{-1} = \varphi(p)$ up to a polynomial order.

Properties of ξ: (exercise)

1) $\xi(p) = \infty$ for $p \geq p_c$.

2) ξ is continuous and non-decreasing on $[0, p_c)$

3) $\xi(0) = 0$ and $\xi(p) \to \infty$ when $p \nearrow p_c$.

Exercise: Let $p > \frac{1}{2}$, show that there is $\psi(p) > 0$ s.t.

$$\mathbb{P}_p[0 \longleftrightarrow \partial\Lambda_n \text{ and } 0 \not\longleftrightarrow \infty] \leq e^{-\psi(p)\cdot n} \quad \text{for large } n.$$

(Hint: blocking circuit)