## 2) Useful Identities & Applications

Probability space $\Omega = \{0,1\}^{\Lambda}$ where $\Lambda$ is finite or countable.

Ex: $\Lambda = E(\mathbb{Z}^d)$, $\Lambda = E([-n,n]^d) =: \Lambda_n$

By abuse of notations, since we look at the bond percolation, we write $\Lambda_n$ for the truncated graph $[-n,n]^d$ and also its edge set (or vertex set).

Partial order on $\Omega$: $\omega \le \omega' \Leftrightarrow \omega(e) \le \omega'(e), \forall e \in \Lambda_n$.

*) A function $f : \Omega \to \mathbb{R}$ is said to be increasing if $\omega \le \omega' \Rightarrow f(\omega) \le f(\omega')$.

*) An event is said to be increasing if $\mathbb{1}_A$ is increasing.

Exercise: Check that the increasing events generate the whole $\sigma$-algebra.

Ex: 1) $x, y \in V(\Lambda)$, $\{x \leftrightarrow y\} := \{$ an open path connects $x$ to $y\}$.

2) $U, V \subseteq V(\Lambda)$, $\{U \leftrightarrow V\} := \bigsqcup_{\substack{x \in U \\ y \in V}} \{x \leftrightarrow y\}$.

3) $x \in V(\Lambda)$, $\{x \leftrightarrow \infty\} := \{x$ belong to an infinite cluster$\}$.

## 2.1) Harris-FKG inequality

**Prop:** (Harris inequality)

*) If $A$ and $B$ are two increasing events, then $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\,\mathbb{P}(B)$.

*) If $f$ and $g$ are two increasing functions in $L^2(\mathbb{P})$, then

$$\mathbb{E}[fg] \geq \mathbb{E}[f]\,\mathbb{E}[g].$$

**Rmk:** 1) The first and the second parts are equivalent.

2) To show the first part, it is enough to check in the case of $\Lambda_n$, then by "approximation" (maybe as an exercise, consequence of $\pi$-$\lambda$ or the monotone class lemma), it gets extended to $\Lambda$.

3) Sometimes it is also called the FKG inequality (Fortuin-Kasteleyn-Ginibre).

In the i.i.d. case (i.e., Bernoulli percolation), it was shown by Harris ('60) then extended to more general measures by F-K-G ('71).

4) The statement rewrites as $\mathbb{P}(B|A) \geq \mathbb{P}(B)$, interpreted as

"When an increasing event holds, it is easier for another increasing event to hold" OR "A and B are positively correlated".

5) We may also define the notion of decreasing events and we get similar identities.

(How do they look like? Prove them.)

Proof: We show the statement for increasing events / functions that depend only on finitely many edges. The proof can be done by induction.

$f(\omega) = f(\omega_1)$, $g(\omega) = g(\omega_1)$

(a) Assume $f, g$ are such functions depending only on $\omega_1 := \omega(e_1)$, the state of the edge $e_1$, then

$$[f(\omega_1) - f(\omega_1')][g(\omega_1) - g(\omega_1')] \geq 0 \quad \text{for any } \omega_1, \omega_1' \in \{0,1\}.$$

$$\Rightarrow \sum_{s,s' \in \{0,1\}} [f(\omega_1) - f(\omega_1')][g(\omega_1) - g(\omega_1')] \, \mathbb{P}_p[\omega_1 = s] \, \mathbb{P}_p[\omega_1' = s'] \geq 0$$

$$\Rightarrow 2\left(\mathbb{E}_p[fg] - \mathbb{E}_p[f]\mathbb{E}_p[g]\right) \geq 0 \quad \Rightarrow \quad \text{OK}.$$

(b) Assume that the statement holds for such functions $f, g$ depending on $k$ edges $e_1, \dots, e_k$.

Let us write $\tilde{\mathbb{P}}$ for the joint distribution of $\{e_1, \dots, e_{k-1}\}$ and $\mathbb{P}_k$ for the distr. of $e_k$.

Then, define $\tilde{f}(\omega_1,\dots,\omega_{k-1}) = \mathbb{E}_k[f(\omega_1,\dots,\omega_{k-1},\omega_k)] \left(= \mathbb{E}[f(\omega) \mid e_1,\dots,e_k]\right)$.

$$= p\, f(\omega_1,\dots,\omega_{k-1},1) + (1-p)\, f(\omega_1,\dots,\omega_{k-1},0),$$

where we integrate the last r.v. out. Similarly, we define $\tilde{g}(\omega_1,\dots,\omega_{k-1})$. We write

$$\mathbb{E}[fg] = \tilde{\mathbb{E}} \otimes \mathbb{E}_k[fg] \geq \tilde{\mathbb{E}}\left[\mathbb{E}_k[f] \cdot \mathbb{E}_k[g]\right] = \tilde{\mathbb{E}}[\tilde{f} \cdot \tilde{g}]$$

$$\overset{\text{Induction}}{\geq} \tilde{\mathbb{E}}[\tilde{f}]\,\tilde{\mathbb{E}}[\tilde{g}] = \mathbb{E}[f] \cdot \mathbb{E}[g] \quad \text{to conclude.} \quad \blacksquare$$

Exercise: (square-root trick)

Given $n$ increasing events $A_1,\dots,A_n$, show that

$$\max\{\mathbb{P}(A_i) : 1 \leq i \leq n\} \geq 1 - [1 - \mathbb{P}(A_1 \cup \dots \cup A_n)]^{1/n}.$$

Application: We want to show that $\theta(\tfrac{1}{2}) = 0$ which implies $p_c \geq \tfrac{1}{2}$.

Proof: Assume that $\theta(\tfrac{1}{2}) > 0$, then by the uniqueness result, we know that

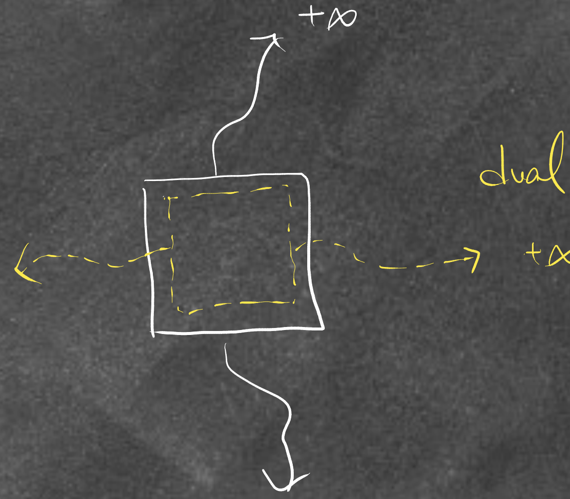
$$\mathbb{P}_{\frac{1}{2}}(\underbrace{\partial \Lambda_n \longleftrightarrow \infty}_{B_n :=}) \xrightarrow[n \to \infty]{} 1.$$

$B_n$

Define $B_n^{top}$, $B_n^{bottom}$, $B_n^{left}$, $B_n^{right}$ for the events that the top/bottom/left/right boundary of $\partial\Lambda_n$ is connected to infinity. By rotation invariance, they have the same prob.

Moreover, $B_n = B_n^{top} \cup B_n^{bottom} \cup B_n^{left} \cup B_n^{right}$, the squre-root trick gives

$$\mathbb{P}(B_n^{top}) \geq 1-(1-\mathbb{P}(B_n))^{1/4} \longrightarrow 1.$$




Take a small $\varepsilon>0$. We can find a large $N$ s.t.

$$\mathbb{P}(B_N^{top}) \geq 1-\varepsilon.$$

Then,

$$\underbrace{\mathbb{P}(B_N^{top} \cap B_N^{bottom} \cap \underbrace{\tilde{B}_N^{left} \cap \tilde{B}_N^{right}}_{\text{events in the dual graph}}}_{C_n :=}) \geq 1-4\varepsilon$$

The event $C_n$ does not depend on the state of the edges in $\Lambda_n$, so

$$\mathbb{P}(C_n \cap \text{all the edges in } \Lambda_n \text{ are closed}) = \mathbb{P}(C_n)\cdot\left(\tfrac{1}{2}\right)^{|E_n|} > 0.$$

If the event on the LHS holds, then $N=2$, giving a contradiction.

$\boxtimes$

Exercise: Show that $p_c \geq \frac{1}{2}$ for the site percolation on the triangular lattice.

## 2.2) Van den Berg–Kesten inequality

The BK inequality is a key step to the proof of $p_c \leq \frac{1}{2}$.

We establish the inequality in this section and we will need the main result from the next section (exp. decay) to prove this fact.

Given an (increasing) event $A$ and a configuration $\omega \in A$, we say that $I \subseteq E(G)$ $= \{e_1, \dots, e_k\}$ is a witness of $A$ for $\omega$, denoted $\omega_I \in A$ if

*) $\omega(e_1) = \dots = \omega(e_k) = 1$.

*) for any configuration $\omega'$ satisfying $\omega'(e_1) = \dots = \omega'(e_k) = 1$, we also have $\omega' \in A$.

We say that two event $A$ and $B$ can be realized disjointly if there exists random witnesses $I = I(\omega)$ and $J = J(\omega)$ of $A$ and $B$ that are disjoint. We write $A \circ B$.

Prop: (BK inequality)

For two increasing events $A, B$ depending only on finitely many edges, we have

$$\mathbb{P}(A \circ B) \leq \mathbb{P}(A)\,\mathbb{P}(B).$$

Rmk: 1) We assume the dependency on finitely many edges but it also holds in our setting of countably many edges.

2) This describes a negative correlation result, which is not surprising. Indeed, if the event $B$ cannot use the edges used by $A$, the prob. that it holds gets smaller.

3) Idea of the proof: the RHS is the prob. if $A$ and $B$ were independent. We construct "auxiliary edges" to recover this quantity.

Proof: Assume that $A$ and $B$ depend only on edges $e_1, \dots, e_n$.

We duplicate these edges to get $e_1, e_1', e_2, \dots, e_n, e_n'$ and consider i.i.d. Ber($p$) on these edges.

$\hookrightarrow$ $(e_k)$ are "old" edges
$(e_k')$ are "new" edges

Write $\hat{\mathbb{P}}$ for this product measure and $\hat{\omega} = (\omega(e_1), \omega(e_1'), \dots, \omega(e_n'))$.

For each $k \leq n$, denote $\omega^k = (\omega(a_1), \dots, \omega(a_{k-1}), \omega(a_k'), \dots, \omega(a_n'))$.

Define $\hat{A}^k = \{\hat{\omega} : \omega^k \in A\}$ and $\hat{B} = \{\hat{\omega} : \omega \in B\}$.

In $\hat{B}$, we only use "old" edges whereas in $\hat{A}^{n+1}, \dots, \hat{A}^1$, we replace one by one the old edges by the new ones. Hence,

$$\hat{\mathbb{P}}(\hat{A}^1 \circ \hat{B}) = \mathbb{P}(A)\,\mathbb{P}(B)$$

$$\hat{\mathbb{P}}(\hat{A}^{n+1} \circ \hat{B}) = \mathbb{P}(A \circ B).$$

It is sufficient to show that $\mathbb{P}(\hat{A}^{k+1} \circ \hat{B}) \leq \mathbb{P}(\hat{A}^{k} \circ B)$ for all $k$.

Fix $1 \leq k \leq n$ and the states of all the edges except for $e_k$ and $e_k'$.

"When we also use new additional edges, we have a higher chance to succeed."

We want to construct a bijective map $s : \{0,1\}^{2n} \to \{0,1\}^{2n}$ s.t.

(a) If $\hat{\omega} \in \hat{A}^{k+1} \circ \hat{B}$, then $s(\hat{\omega}) \in \hat{A}^{k} \circ B$

(b) $s$ is measure-preserving, i.e. $\hat{\mathbb{P}}(\hat{\omega}) = \hat{\mathbb{P}}(s(\hat{\omega}))$.

If $\hat{\omega} \notin \hat{A}^{k+1} \circ \hat{B}$, we simply let $s(\hat{\omega}) = \hat{\omega}$.

If $\hat{\omega} \in \hat{A}^{k+1} \circ \hat{B}$ and let $I$ and $J$ to be disjoint witnesses of $\hat{A}^{k+1}$ and $\hat{B}$.

If $e_k \notin I$, set $s(\hat{\omega}) = \hat{\omega}$.

If $e_k \in I$, set $s(\hat{\omega})$ by exchanging the values of $\omega(e_k)$ and $\omega(e_k')$.

$\Rightarrow$ (a) and be are clearly satisfied.

$\blacksquare$

Exercise: 1) Find an example of increasing events $A$ and $B$ s.t. $\mathbb{P}(A \circ B) < \mathbb{P}(A)\,\mathbb{P}(B)$.

2) (a) Write $R_n = [0,n] \times [0,n-1]$ and $\mathcal{H}_n$ for the event $\{$horizontal crossing in $R_n\}$.

Use the fact that $\mathbb{P}_{\frac{1}{2}}(\mathcal{H}_n) = \frac{1}{2}$ to show $\mathbb{P}_{\frac{1}{2}}(0 \longleftrightarrow \partial\Lambda_n) \geq \frac{1}{2n}$.

(b) Let $\mathcal{H}_n' = \{$horizontal SAW in $R_n\}$. What can we say for $\mathbb{P}(\mathcal{H}_n')$?

(c) Let $C_n(x) = \{x \longleftrightarrow x + \partial\Lambda_n\}$. Show that $\mathbb{P}_{\frac{1}{2}}(C_n(0) \circ C_n(0)) \geq \frac{1}{4n}$.

What can we say for $\mathbb{P}_{\frac{1}{2}}(C_n(0))$?

2.3) Russo's identity

Goal: We want to study the behavior of $\mathbb{P}_p(A)$ when $p$ varies (one can reasonably assume that $A$ is increasing).

An edge $e \in E$ is said to be pivotal for $A$ and $\omega$ if $\omega_e \notin A$, $\omega^e \in A$,

where $\omega^e / \omega_e$ stand for the configurations with $\omega(e) = 1$ / $\omega(e) = 0$.

Rmk: $\omega(e)$ decides whether $\omega \in A$ or $\omega \notin A$ and the value of $\omega(e)$ is independent of the event "$e$ is pivotal for $A$".

Write $N(A) := N(A, \omega) = \#$ pivotal edges for $A$ and $\omega$.

<u>Prop</u>: (Russo's identity) If $A$ is an increasing event depending on <u>finitely</u> <u>many</u> edges, then $\frac{d}{dp}\mathbb{P}_p(A) = \mathbb{E}_p[N(A)]$.

<u>Proof</u>: We couple the percolation measures $(\mathbb{P}_p)_{p\in[0,1]}$ using uniform r.v.s.

Let $e_1,\dots,e_n$ be the only edges that $A$ depends on. Consider $X_1,\dots,X_n$ i.i.d. uniform random variables on $[0,1]$. For each $\bar{p} \in [0,1]^n$, define

$\omega_{\bar{p}} = (\mathbb{1}_{X_1\le p_1},\dots,\mathbb{1}_{X_n\le p_n})$. Fix $\bar{p} = (p_1,\dots,p_n)$ and consider

$\bar{p}' = (p_1',\dots,p_n')$ where there exists $1\le j\le n$ s.t. $p_j' > p_j$ and $p_i' = p_i$ for all $i\neq j$.

Then,
$$\begin{aligned}
\mathbb{P}_{\bar{p}'}(A) - \mathbb{P}_{\bar{p}}(A) &= \mathbb{E}[\mathbb{1}_{\omega_{\bar{p}'}\in A} - \mathbb{1}_{\omega_{\bar{p}}\in A}] \qquad (\omega_{\bar{p}} \le \omega_{\bar{p}'}) \\
&= \mathbb{P}[\omega_{\bar{p}'}\in A \text{ but } \omega_{\bar{p}}\notin A] \\
&= \mathbb{P}[e_j \text{ is pivotal for } A \text{ and } X_j \in (p_j, p_j')] \\
&= (p_j' - p_j)\cdot \mathbb{P}[e_j \text{ is pivotal for } A]
\end{aligned}$$

$$\Rightarrow \quad \frac{d}{dp_j}\mathbb{P}_{\bar{p}}(A) = \mathbb{P}_{\bar{p}}[e_j \text{ is pivotal for } A]$$

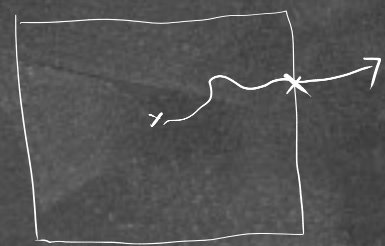
After summing over $j$, we find $\frac{d}{dp}\mathbb{P}_p(A) = \sum_{e\in E} \mathbb{P}_p(e \text{ is pivotal for } A)$

$$= \mathbb{E}_p[N(A)] \,.$$

$\boxed{///}$

<u>Exercise</u>: For a finite edge set and <u>any</u> function $F:\{0,1\}^E \to \{0,1\}$, use a direct computation to show that
$$\frac{d}{dp} f(p) = \frac{1}{p(1-p)} \sum_{e\in E} \mathrm{Cov}(F, \omega_e) \quad , \quad f(p) := \mathbb{E}_p[F] .$$
Deduce the Russo's identity from this.

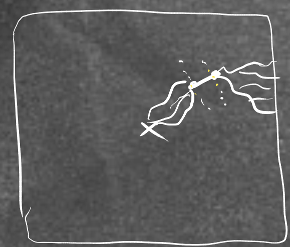
In the next section, we will estimate the quantity $\mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$ for $p<p_c$ and show that it decreases exponentially.




Let $A_n = \{0 \leftrightarrow \partial\Lambda_n\}$. What does it mean for an edge $e=\{x,y\}$ to be pivotal for $A$ and $\omega$?

(a) $0 \leftrightarrow x \quad , \quad y \leftrightarrow \partial\Lambda_n$

(b) if $e$ is closed then $A$ does not hold.



We have $\frac{d}{dp}\mathbb{P}_p(A_n) = \mathbb{E}_p[N(A_n)] \geq \mathbb{E}_p[N(A_n)\mathbb{1}_{A_n}] = \mathbb{P}_p(A_n)\cdot\mathbb{E}_p[N(A_n)\mid A_n]$.

$$\Rightarrow \quad \frac{d}{dp}\ln\mathbb{P}_p(A_n) \geq \mathbb{E}_p[N(A_n)\mid A_n]$$

$$\Rightarrow \quad \frac{\mathbb{P}_{p_1}(A_n)}{\mathbb{P}_{p_2}(A_n)} \leq \exp\left\{-\int_{p_1}^{p_2}\mathbb{E}_p[N(A_n)\mid A_n]\,dp\right\} \quad \text{for } p_1<p_2.$$

If for $p_1<p_2<p_c$ we can find $c>0$ s.t. $\mathbb{E}_p(N(A_n)\mid A_n] \geq cn$ for $p\in(p_1,p_2)$,

then we can conclude $\frac{\mathbb{P}_{p_1}(A_n)}{\mathbb{P}_{p_2}(A_n)} \leq \exp(-(p_2-p_1)\cdot c\cdot n) \Rightarrow$ exp. decay. $\square$