1) Basic Properties of the Bernoulli Percolation

Recall the notation of $\mathbb{P}_p$ and ω_p on $G = \mathbb{Z}^d$ or some "nice" graph.

1.1) Coupling

Intuitively, when p gets larger, the clusters get bigger as well.

Goal: We want to make this idea mathematically rigorous.

Given $p \le p' \in [0,1]$, how to compare $X \sim \mathrm{Ber}(p)$, $X' \sim \mathrm{Ber}(p')$?

We consider $\mathcal{U} \sim \mathrm{Unif}([0,1])$ and define $Y = \mathbb{1}_{u \le p}$, $Y' = \mathbb{1}_{u \le p'}$

then $Y \le Y'$ a.s. and $X \overset{(d)}{=} Y$, $X' \overset{(d)}{=} Y'$.

This is called a coupling between random variables. We change the underlying probability space so that we can compare directly the values of r.v.s. without modifying their marginal distributions. Note that we lose independence between the r.v.s.

Exercise: Construct a coupling between ω and ω' where $\omega \sim \mathbb{P}_p$ and $\omega' \sim \mathbb{P}_{p'}$ are Bernoulli percolations on the same graph with different parameters $p \leq p'$.

Rmk: We will always assume that the measures $(\mathbb{P}_p)_{p\in[0,1]}$ are defined this way throug a coupling.

Exercise: Given a marked point $0 \in V(G)$ (called the origin).

Deduce that
$$\theta : [0,1] \to \mathbb{R}$$
$$\left\{ p \mapsto \mathbb{P}_p(\, 0 \text{ belongs to an } \infty \text{ cluster}\,) \right.$$

is a non-decreasing function.

Note that on "1-periodic" graphs, θ does not depend on the choice of 0.

Define $\quad p_c = \sup\{\, p \in [0,1] : \theta(p) = 0 \,\}$.

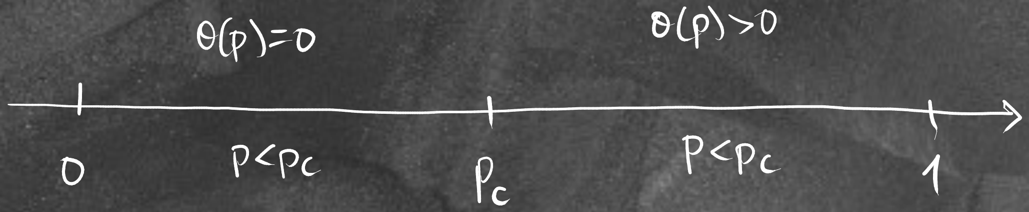
<u>Exercise</u>: Check the following properties:

1) The function $p \mapsto \theta(p)$ is right-continuous on $[0,1]$.

2) The function $p \mapsto \theta(p)$ is left-continuous on $(p_c, 1)$.

(Hint: use the uniqueness)

3) Show that $p \mapsto \theta(p)$ is strictly increasing on $(p_c, 1]$.



<u>Questions</u>: 1) Where is p_c? $p_c \neq 0$? $p_c \neq 1$? (Triviality of the P.T.)

2) $\theta(p_c) > 0$ or $\theta(p_c) = 0$?

We will see that $\theta(p_c) = 0$ for $\mathbb{Z}^2$. It is conjectured that $\theta(p_c)=0$ for all $\mathbb{Z}^d$ but only know for $d \geq 11$ ('90, Hara-Slade, $d \geq 19$,

'15, Fitzner-van der Hofstad, $d \geq 11$)

$\rightsquigarrow$ lace-expansion should work in the same way for $d \geq 6$.

Exercise:

1) Show that on $\mathbb{Z}^d$, $p_c \geq \frac{1}{2d}$ or even $p_c > \frac{1}{2d}$. (Hint: G-W.)

2) Show that on $\mathbb{Z}^d$, $p_c < 1$. (Hint: look at $\mathbb{Z}^2$ + dual graph)

3) Show that on a more general graph G, $p_c \geq \frac{1}{\mu}$

where μ is the connective constant of G. (Hint: SAW)

1.2) Uniqueness of the Infinite Cluster

We consider a periodic lattice such as $\mathbb{Z}^d$, planar triangular / hexagonal lattice on which we define a bond / site Bernoulli percolation.

Thm: For a fixed $p \in [0,1]$, we have

- either a.s. there is no infinite cluster
- or a.s. there is a unique infinite cluster

Proof: We decompose the proof in 3 steps.

1) Show that if an event A is invariant under a vector $e \neq 0$, then $P(A) = 0$ or 1. $\rightsquigarrow$ Kolmogorov or "approximation method"

$\Rightarrow$ Thus, the number of clusters N is almost surely a constant.

2) Exclude the case $N = k$ a.s. for $k \geq 2$, $k \neq \infty$ using the "finite energy property".

3) Look at "trifurcation" points to exclude the case $N = \infty$ a.s.

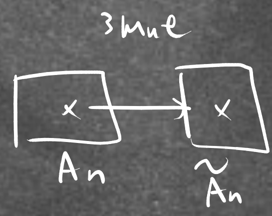
1) Claim (exercise): any event A can be "approximated" by a sequence of events that depend only on a finite number of edges. In other words, we can find a sequence of events $(A_n)_{n\geq 0}$ s.t. A_n depends only on a finite number of edges for all $n \geq 0$ and $\mathbb{P}(A \triangle A_n) \xrightarrow[n\to\infty]{} 0$

where $A \triangle A_n = (A \setminus A_n) \sqcup (A_n \setminus A)$.

Given an event A which is invariant under a vector $e \neq 0$.

Consider $(A_n)_{n\geq 0}$ as above and a sequence $(m_n)_{n\geq 0}$ s.t. A_n depends only on edges in $[-m_n, m_n]^d$.

Define the shift operator $s: \begin{cases} \Omega \to \Omega \\ \omega \mapsto \omega(\cdot - e) \end{cases}$.



The assumption states that $sA = A$.

For each $n \geq 0$, define $\tilde{A}_n = s^{3m_n}(A_n)$, which is the event A_n shifted by $3m_n$, so depends only on the edges in $[-m_n, m_n]^d + 3m_n e$.

$\Rightarrow$ A_n and $\tilde{A}_n$ are independent.

$$\Rightarrow \quad \mathbb{P}(A_n \cap \tilde{A}_n) = \mathbb{P}(A_n)\,\mathbb{P}(\tilde{A}_n) \longrightarrow \mathbb{P}(A)^2 .$$

At the same time, we also have

$$\mathbb{P}(A \triangle (A_n \cap \tilde{A}_n)) \leq \mathbb{P}(A \triangle A_n) + \mathbb{P}(A \triangle \tilde{A}_n) \longrightarrow 0 .$$

Hence, $\mathbb{P}(A) = \mathbb{P}(A)^2 \Rightarrow \mathbb{P}(A) \in \{0,1\}$.

For any $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, the event $\{N = k\}$ is invariant under any vector $e \neq 0$. So we can find a $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ s.t.

$N = k$ a.s.

2) Let $k \geq 2$, $k \neq \infty$ is s.t. $N = k$ a.s. We want to show that it is not possible.

The event $B = \{N = k\}$ can be approximated by

$n \geq 0$, $B_n = \{N = k$ and the k infinite clusters intersect $[-n,n]^d\}$.

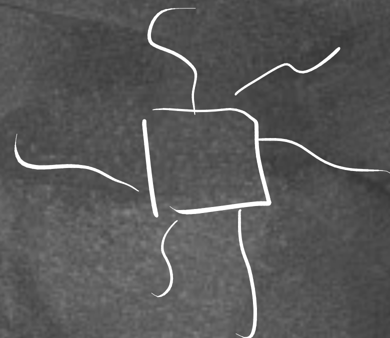
Fix $n \geq 0$ large enough so that $\mathbb{P}(B_n) > 0$.

Consider another event $C = \{$ all infinite clusters touch $[-n,n]^d\}$

and we can easily deduce that $\mathbb{P}(C) \geq \mathbb{P}(B_n) > 0$.

Note that it is not the case for B_n!

Note that the event C depends only on the edges <u>outside</u> of $[-n,n]^d$ so



$$\mathbb{P}\Big(C \cap \underset{\text{all open}}{\boxed{\ }}\Big) \overset{\perp}{=} \mathbb{P}(C)\,\mathbb{P}\Big(\underset{\text{all open}}{\boxed{\ }}\Big) > 0,$$ which means that

$\mathbb{P}(N=1) > 0$. But since $\mathbb{P}(N=k)=1$ with $k \geq 2$, $k \neq \infty$, it is contradictory. Hence, we reach at the conclusion

$N=0$ a.s. OR $N=1$ a.s. OR $N=\infty$ a.s.

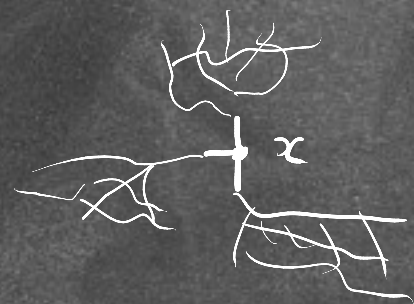
3) Now, we want to show that it is impossible to have $\mathbb{P}(N=\infty)=1$. We may assume that $0<p<1$.

A vertex $x \in \mathbb{Z}^d$ is said to be a trifurcation if

*) x is in an ∞ cluster

*) exactly 3 adjacent edges of x are open

*) if the adjacent open edges of x are closed, then we obtain three ∞ clusters.



Assume that $\mathbb{P}(N=\infty)=1$. Define

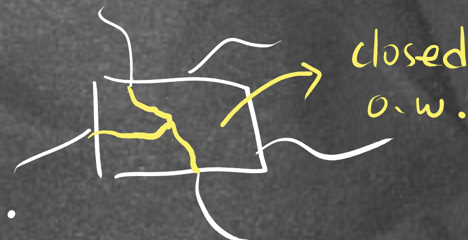
$\forall n \geq 1, \quad D_n = \{$ at least 3 ∞ clusters intersect $[-n,n]^d \}$.

We can find a large $n \geq 1$ s.t. $\mathbb{P}(D_n) > 0$.

Due to the independence, we can "resample" the state of the edges in $[-n,n]^d$, or rather, we can choose a configuration in $[-n,n]^d$ so that the origin 0 is a trifurcation. We proceed as follows.

(a) Note that $[-n,n]^d$ is a finite graph. Any configuration of edges has at least probability $\min(p,1-p)^{\#\text{edges in } [-n,n]^d} > 0$.

(b) For each configuration $\omega \in D_n$, we can choose a "nice" configuration inside $[-n,n]^d$ s.t. 0 is a trifurcation.



$\Rightarrow \quad c := \mathbb{P}(0 \text{ is a trifurcation}) \geq \mathbb{P}(D_n) \cdot \min(p,1-p)^{\#\text{edges}} > 0$.

Let us study the structure of trifurcation points. $\rightsquigarrow$ tree structure

For each trifurcation point x, define 3 ∞ open self-avoiding paths from x. Check that:

*) The union of trifurcation points with the open SA paths forms a forest.

*) An intersection of SA paths with $[-n,n]$ is called a node.

We have the relation $\quad \#\text{nodes} \geq 2\,\#\text{trifurcation points}$.

Define the r.v.s $X_n = \#\text{ trifurcations in } [-n,n]$

$$= \sum_{x \in V_n} \mathbb{1}_{x \text{ is a trifurcation}}$$

and $Y_n = \#\text{ nodes}$, then we should find $\mathbb{E}[Y_n] \geq 2 \cdot \mathbb{E}[X_n]$.

Moreover, $\mathbb{E}[Y_n] \sim \text{cst} \cdot n^{d-1}$, $\mathbb{E}[X_n] \sim \text{cst} \cdot c \cdot n^d$.

There is a contradiction for large n. $\blacksquare$

Exercise: 1) For which more general graphs does this theorem hold?

2) If we do not require uniqueness, to which graph can this theorem be extended?

Exercise: What happens if we look at the tree $\mathbb{T}_d$?

Exercise: Check the following properties

For $p \in [0,1]$, $\theta(p) > 0 \iff \mathbb{P}_p(N \geq 1) = 1$.

1.3) Ideas towards $p_c = \frac{1}{2}$.

We will show that for $\mathbb{Z}^2$, $p_c = \frac{1}{2}$ and $\theta(p_c) = 0$.

Heuristics: Define the dual graph of $\mathbb{Z}^2$ as follows: $G = \mathbb{Z}^2$, $G^* =$ dual graph of G.

$V(G^*) = F(G)$, $E(G^*) \cong E(G)$ where two adjacent faces of G become two adjacent vertices of G^*.

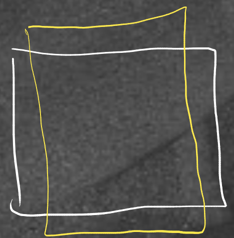
To a configuration ω on $E(G)$ we associate its dual configuration ω^* on $E(G^*)$ as follows:

$$\omega^*(e^*) = 1 - \omega(e) \implies \text{if } \omega \sim \mathbb{P}_{p,G} \text{ then } \omega^* \sim \mathbb{P}_{1-p,G^*}.$$

<u>Key property</u>: The dual of $\mathbb{Z}^2$ is $(\mathbb{Z}+\frac{1}{2})^2 \cong \mathbb{Z}^2$. It is self-dual.

Heuristically, if under $\mathbb{P}_p$, there is a huge / infinite cluster, then under $\mathbb{P}_{1-p}$, the clusters need to be small, so p_c must be $\frac{1}{2}$.

More precisely, if we look at $p=\frac{1}{2}$ and a finite rectangle $R_n = [0,n] \times [0,n+1]$.



The "dual" of R_n is also $R_n^* \cong R_n$.

Moreover, there is a horizon crossing in R_n <u>iff</u> there is no vertical crossing in R_n^*.

Rigorous Part

$\Rightarrow \quad \mathbb{P}_{\frac{1}{2}}(\quad \text{in } R_n) + \mathbb{P}_{\frac{1}{2}}(\quad \text{in } R_n^*) = 1.$

But since the two quantities are equal, they are both $\frac{1}{2}$.

In consequence, for $p<\frac{1}{2}$, the horizontal crossing has small probabilities, so $p_c \leq \frac{1}{2}$,
$p>\frac{1}{2}$, ———— large probabilities, so $p_c \geq \frac{1}{2}$.

Need to be made more precise.

We will introduce some useful inequalities later to make this rigorous.