0) (Planar) Statistical Mechanics

0.1) Motivation

We want to describe the (macroscopic) behaviors of a system composed of a huge number of (microscopic) particles. Theoretically, if the behavior of <u>each</u> particle can be described, then one should be able to say something about the system. However, the number of particles is astronomical (1 mole $\approx 10^{23}$) and it is costly (even impossible) to solve the system.

Boltzmann (end 19th century) gives a probabilistic formalism.

Ω = sample space (discrete, countable, possibly infinite)

$H : \Omega \to \mathbb{R}$ energy function (Hamiltonian)

$\beta = \frac{1}{T}$ inverse temperature

$\Rightarrow$ Boltzmann distribution $\mathbb{P}_\beta(\omega) = \frac{1}{Z_\beta} \exp(-\beta H(\omega))$

where $Z_\beta = \sum_{\omega \in \Omega} \exp(-\beta H(\omega))$ is called

partition function.

Interpretations :

*) At a fixed temperature, an exponential weight is associated to each configuration.

*) At high temperature ($\beta \to 0^+$), the weights are almost a constant function of configurations, so each state is equiprobable.

*) At low temperature ($\beta \to +\infty$), the weights concentrate on minimizers of the energy function H.

$\Rightarrow$ This corresponds to the intuitions we have in physics (thermodynamics ?)

Rmk: 1) $Z_\beta < \infty$ if Ω is finite, so P_β is well defined.

2) If Ω is infinite, then Z_β is not always finite. The finiteness of Z_β may depend on β.

0.2) First model (polymer / self-avoiding walk)

Consider the square lattice $\mathbb{Z}^2$ (or any other regular lattice)

Polymer = self-avoiding walk from the origin

SAW = a path that does not intersect its trajectory.

(Paul Flory, chemist, 1953)

Given a SAW ω, write $H(\omega) = |\omega|$ = length of the walk.

Then, $$Z_\beta = \sum_\omega \exp(-\beta |\omega|) = \sum_{N \geq 1} \lambda_N \exp(-\beta N)$$

where λ_N denotes the number of SAWs of length N.

Here, the sample space $\Omega = \{$ all SAWs $\}$ is countably infinite.

To make the model well defined, we need to ensure that $Z_\beta < \infty$.

Clearly, for $\beta = 0$, $Z_\beta = \infty$. We need to understand the asymptotic behavior of λ_N (exp. growth) so that we can find β for which Z_β is finite.

Observation: for any $n, m \geq 1$, if ω is a SAW of length $n+m$, then $(\omega_1, \dots, \omega_n)$ and $(\omega_{n+1}, \dots, \omega_{n+m})$ are SAWs of length n and m.

The subadditive lemma (exercise) says that $\mu := \lim (\lambda_N)^{1/N} \in [1, \infty)$

and that $\lambda_N \geq \mu^N$ for all $N \geq 1$.

μ is called the connective constant (of the graph).

Let $\beta_c = \ln(\mu)$, then $Z_\beta < \infty \iff \beta > \beta_c$.

Exercise: Check the following:

1) In $\mathbb{Z}^2$, $2 < \mu < 3$

2) In $\mathbb{Z}^3$, $\mu > 3$

3) In the planar triangular lattice, $\mu > 3$.

4) In the planar hexagonal lattice, $\mu < 2$

5) For $\mathbb{Z} \times \{0,1\}$, $\mu = \frac{1}{2}(1+\sqrt{5})$.

Rmk:

1) It is highly non trivial to compute the connective constant of a graph. The only non trivial graph for which this value can be computed is for the planar hexagonal lattice where $\mu = \sqrt{2+\sqrt{2}}$.

(H. Duminil-Copin, S. Smirnov, 2010)

2) Although μ depends on the lattice we consider, it is conjectured that

$$Z_\beta = (\beta - \beta_c)^{-\gamma + o(1)} \quad \text{when} \quad \beta \to \beta_c^+ \quad \text{for some } \gamma > 0$$

that only depends on the dimension of the lattice.

For example, when $d=2$, it is conjectured that $\gamma = \frac{43}{32}$.

3) Heuristically, if $\lambda_N \sim \mu^N \cdot N^\alpha$ then $Z_\beta \sim (\beta - \beta_c)^{-(1+\alpha)}$.

Thus, (conjecture) $\lambda_N = \mu^N \cdot N^{11/32 + o(1)}$.

Exercise: How to sample such a SAW? Think of a Markov process whose invariant measure is $\mathbb{P}_\beta$ $(\beta > \beta_c)$.

0.3) Bernoulli percolation

Consider a graph with finite / countably many vertices and assume it has bounded degrees. Write $G = (V, E)$. (Ex: $\mathbb{Z}^2$)

Bond percolation: Given $p \in [0,1]$. Consider a family of i.i.d. r.v.s $(\omega(e))_{e \in E}$ s.t. $\omega(e) \sim \mathrm{Ber}(p)$.

If $\omega(e) = 1$, we say that e is open (we draw it on the board).

If $\omega(e) = 0$, we say that e is closed (we don't draw it).

We get a random subgraph from ω which is given by open edges.

By abuse of notation, we still write ω for the subgraph.

Site percolation: The same definition with the difference that the r.v.s are given on vertices.

Notation: $\mathbb{P}_p$ = the Bernoulli percolation measure with parameter p

ω_p = a realization of $\mathbb{P}_p$ / $\omega_p \sim \mathbb{P}_p$.

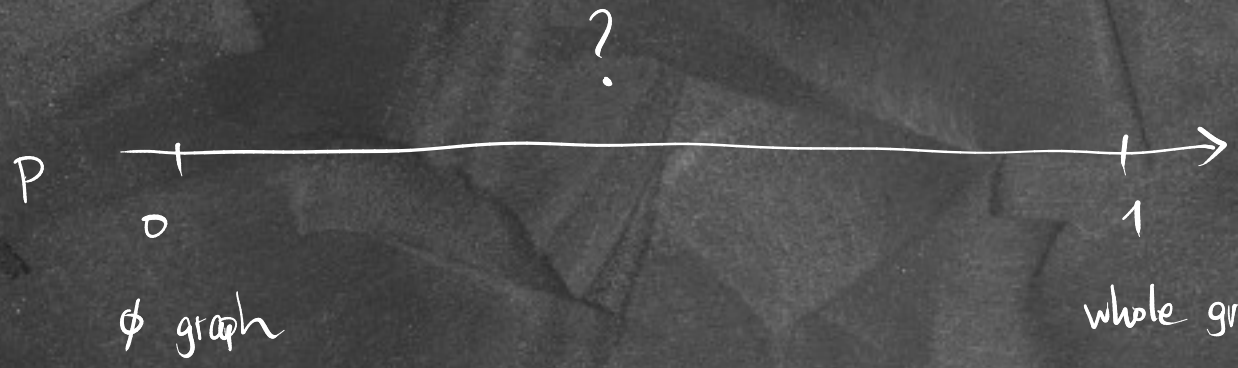
Rmk: The terminology "Bernoulli percolation" is used for such a random subgraph model where edges / vertices are i.i.d. Bernoulli r.v.s. (independence!)

Generally speaking, without independence, we may still call it a percolation model, e.g. the random-cluster model we will see towards the end of this series of lectures.

Rmk: The independence between edges makes the "infinite-volume" measure more straightforward and we are not bothered by the "well-definiteness" or the "uniqueness" of the infinite-volume measure(s).

Exercise: Show that a bond percolation is equivalent to a site percolation (on a modified graph). How about its converse? Construct some examples.

Question: Understand the behavior of the random subgraph ω when p varies. For example, is there an infinite cluster? Is it unique? How big are the clusters if they are finite? Etc.



Exercise: Let $\mathbb{T}_d$ be the d-regular tree, i.e. an infinite connected graph with countably many vertices without cycles where all the vertices are of degree d. Ex: $\mathbb{T}_2 = \mathbb{Z}$, $\mathbb{T}_3 =$  .

Exercise: Define the random-cluster model and discuss the following:

1) Why it can only be defined on a finite graph?

2) The heuristic behavior of the model when q varies.

($q \to 0^+$ UST, $q=1$ Bernoulli percolation,

$q>1$ configurations with more clusters are preferred

$q<1$ configurations with less clusters are preferred)

3) The definition of the model is "global", i.e. one needs to count the number of clusters so needs to know where a configuration looks like in the <u>whole</u> graph.

4) For the planar random-cluster model, the FK-loop representation makes the definition "local". (This may be hard to discuss at the moment, maybe I can mention it later in the lecture.)